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SOME RESULTS ON COMPLEX INTERPOLATION OF T_q^p SPACES *

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0. Introduction and preliminary notations

The class of "tent spaces" T_q^p was introduced by Coifman, Meyer and Stein in [6] and [7], where applications to Harmonic Analysis were given. Recently several papers have paid attention to these spaces. See [1], [2], [3], [4], [9] and the references given there.

The definition of tent spaces is made using two kinds of functionals. Let $0 < q < \infty$. For a complex-valued measurable function f over $\mathbf{R}_+^{n+1}$, define:

$$A_q f(x) = \left(\int_{\Gamma(x)} |f(y,t)|^q \frac{dy\,dt}{t^{n+1}} \right)^{1/q},$$

and

$$C_q f(x) = \sup \left(\frac{1}{|B|} \int_{\hat{B}} |f(y,t)|^q \frac{dy\,dt}{t} \right)^{1/q},$$

where $\Gamma(x)$ stands for the cone of all $(y,t) \in \mathbf{R}_+^{n+1}$ such that $x \in B(y,t)$ and the "sup" in the definition of $C_q f(x)$ is taken over all balls B that contain the point $x \in \mathbf{R}^n$. For any such ball B, $\hat{B}$ stands for the "tent over B" of all $(y,t) \in \mathbf{R}_+^{n+1}$ with $B(y,t) \subset B$. It is also defined

$$A_\infty f(x) = \sup_{(y,t)\in\Gamma(x)} |f(y,t)|,$$

the non-tangential maximal function.

With the previous notations, T_q^p spaces are defined by the conditions $A_q f \in L^p(\mathbf{R}^n)$ if $0 < p,q < \infty$ and $C_q f \in L^\infty(\mathbf{R}^n)$ if $0 < q < \infty$ and $p = \infty$. If $0 < p < \infty$, T_∞^p is defined to consist of all continuous f with non-tangential boundary limits such that $A_\infty f \in L^p(\mathbf{R}^n)$.

Supplied with the obvious functionals, T_q^p are quasi-Banach spaces. Moreover they are Banach spaces when $1 \leq p,q \leq \infty$.

An important result is the duality theorem. The bilinear map

$$\langle f, g \rangle = |B(0,1)| \int_{\mathbf{R}_+^{n+1}} f(y,t)\, g(y,t) \frac{dy\,dt}{t} \tag{1}$$

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realizes $T_{q'}^{p'}$ as an equivalent dual to T_q^p when $1 \le p < \infty$, $1 < q < \infty$ and p' and q' are the conjugate exponents to p and q. See [7].

The purpose of this paper is to study the complex interpolation of these spaces making use of the fact that there is an isometry from T_q^p into certain vector-valued L^p spaces if $0 < p, q < \infty$. We will obtain the identification

$$(T_{q_0}^{p_0}, T_{q_1}^{p_1})_{[\theta]} = T_q^p \tag{2}$$

when $0 < p_0, p_1, q_0, q_1 < \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Also, we will make a few remarks on the extreme cases. We observe that this identification has been made so far if $q_0 = q_1 = 2$ and $1 \le p_0, p_1 \le \infty$ in [7]; when $q_0 = q_1 = \infty$ and $1 < p_0, p_1 < \infty$ in [1]; when $q_0 = q_1 = 2$ and $0 < p_0, p_1 \le \infty$ in [2] and when $1 < p_0, p_1, q_0, q_1 < \infty$ in [4].

Since the possibility that the parameters could be smaller than one is considered, we will discuss briefly in section 1 the complex interpolation theory for quasi-Banach spaces. Section 2 is devoted to the study of (2) in the intermediate cases $0 < p_j, q_j < \infty$, $j = 0, 1$ and section 3 is devoted to obtain some extreme cases.

1. Complex interpolation theory for quasi-Banach spaces

Complex interpolation theory for quasi-Banach spaces has been considered by several authors and the method which will be briefly described here for the shake of completeness is that of [4], where comments relating it to other possibilities are also given.

The starting point is a complex interpolation pair (X_0, X_1) such that the sum space $X_0 + X_1$ can be continuously embedded into an A-convex space $\mathcal{U}$. A complex quasi-Banach space $\mathcal{U}$ is said to be A-convex, Kalton [11], if whenever ϕ is an $\mathcal{U}$-valued continuous function on the closed unit disk which is analytic on its interior, then

$$\|\phi(0)\| \le C \sup_{|z|=1} \|\phi(z)\|,$$

for some constant C independent of ϕ. This is equivalent to the existence of an equivalent plurisubharmonic quasi-norm on $\mathcal{U}$.

From this point, the idea is to replace in the original construction of Calderón [5] the sum space by the space $\mathcal{U}$. More precisely, the admissible functions in the construction of the complex method will be defined in the closed unit strip $\bar{S}$, will take values in $\mathcal{U}$ and will be bounded, continuous and analytic on S. Furthermore, they will verify the habitual boundary conditions, so that $f(j+it) \in X_j$, $j = 0, 1$, boundedly and continuously. We impose a further condition to a function to be admissible: there must be a sequence of finite rank functions

$$\phi_n : \bar{S} \to X_0 \cap X_1$$

bounded, continuous, analytic on S such that ϕ_n converges to f X_j-uniformly over $j + i\mathbf{R}$, $j = 0, 1$ (thus $\mathcal{U}$-uniformly over $\bar{S}$). It would be interesting to know if this last condition is a consequence of the previous ones, as in the Banach space case. We will denote by $\mathcal{G} = \mathcal{G}(X_0, X_1)$ the class of the above described finite rank functions and by

$\mathcal{F} = \mathcal{F}(X_0, X_1) = \mathcal{F}(X_0, X_1; \mathcal{U})$ the class of all admissible functions. The space $\mathcal{F}$ is a quasi-Banach space when endowed with the functional

$$\|f\|_{\mathcal{H}(\mathcal{U})} = \max\left\{\sup_{t\in\mathbf{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbf{R}} \|f(1+it)\|_{X_1}\right\},$$

where the subscript "$\mathcal{H}(\mathcal{U})$" is placed to follow the notations of [4]. For $0 \le \theta \le 1$ the space of all $f(\theta)$, with admissible f, will be denoted by $X_{[\theta],\mathcal{U}}$ or $(X_0, X_1)_{[\theta],\mathcal{U}}$. It turns out to be a quasi-Banach space when endowed with the habitual quotient quasi-norm. The so defined space $(X_0, X_1)_{[\theta],\mathcal{U}}$ densely contains the intersection $X_0 \cap X_1$ and is continuously embedded in $\mathcal{U}$. It is false in general that it is contained in the sum $X_0 + X_1$. Nevertheless, if we have another pair (Y_0, Y_1) and an A-convex containing space $\mathcal{V}$ for this second pair, then any continuous linear operator $T: \mathcal{U} \to \mathcal{V}$ such that $T: X_j \to Y_j$ with constant M_j, $j = 0, 1$, must map $X_{[\theta],\mathcal{U}}$ into $Y_{[\theta],\mathcal{V}}$ with constant not exceeding $M_0^{1-\theta} M_1^{\theta}$.

The fact that the operator T should be a priori defined and be continuous on the whole space $\mathcal{U}$ is a negative property of this method. However, if (X_0, X_1) has an A-convex containing space, there always exists another A-convex space that will generically be denoted by A (see [4] or [11] for details) such that, whenever an operator T is bounded from X_j into Y_j, $j = 0, 1$, it automatically extends to a bounded operator

$$T_A : A \to \mathcal{V},$$

the preceeding appies and

$$T_A : (X_0, X_1)_{[\theta],A} \to (Y_0, Y_1)_{[\theta],\mathcal{V}}.$$

This will make it important to compute in the quasi-Banach case the complex interpolation spaces with respect to this canonical A-convex containing space, which is called the A-convex envelope in [4].

Another phenomenon that should be mentioned in this summary is that different A-convex containing spaces can give rise to different complex interpolation spaces. On the other hand, if $\mathcal{U}_1$ and $\mathcal{U}_2$ are two A-convex containing spaces for a pair (X_0, X_1) and there is a Hausdorff topological vector space $\mathcal{A}$ in which both of them are continuously contained (in the sense that the elements of the sum space $X_0 + X_1$ are mapped to the same points of $\mathcal{A}$ by both inclusions), then the spaces $(X_0, X_1)_{[\theta],\mathcal{U}_1}$ and $(X_0, X_1)_{[\theta],\mathcal{U}_2}$ are the same. Thus for example, all the A-convex containing spaces that are subspaces of the space of tempered distributions are equivalent for this interpolation method. Also if $\mathcal{U}_1 \subset \mathcal{U}_2$ the corresponding interpolation spaces are the same for $\mathcal{U}_1$ and $\mathcal{U}_2$. Finally, if the sum space is A-convex, for example in the Banach space case, the interpolation spaces are subsets of the sum and all the A-convex containing spaces are equivalent.

Finally we point out that in [4] it is shown that typical examples like couples of l^p, L^p, H^p or T_q^p spaces have A-convex containers and therefore their interpolation lies within the scope of this theory.

2. Identification of the interpolation spaces in the intermediate cases

The first idea for dealing with the couples $(T^{p_0}_{q_0}, T^{p_1}_{q_1})$ is that the tent spaces T^p_q, $0 < p < \infty$, $0 < q \leq \infty$, are subspaces of vector-valued L^p spaces. More precisely: if L^p denotes the ordinary L^p space on $\mathbf{R}^n$ with respect to the Lebesgue measure and L^q_* denotes the L^q space over $\mathbf{R}^{n+1}_+$ with respect to the measure $\frac{dy\,dt}{t^{n+1}}$, the operator T which maps a function $f\colon \mathbf{R}^{n+1}_+ \to \mathbf{C}$ into a function over $\mathbf{R}^n$ whose value at $x \in \mathbf{R}^n$ is the function $Tf(x)$ defined by

$$Tf(x)(y,t) = f(y,t)\chi_{\Gamma(x)}(y,t)$$

is an isometry from T^p_q into $L^p(L^q_*)$, if $0 < p < \infty$ and $0 < q \leq \infty$. See [4] for details about the measurability of Tf. We will prove later that, for $1 < p, q < \infty$, T^p_q are actually complemented subspaces of $L^p(L^q_*)$. This result may be useful not only in the identification of the interpolation spaces, but also on the quick proof of some properties of T^p_q spaces from the corresponding ones of $L^p(L^q_*)$. For example, it follows that T^p_q is reflexive whenever $1 < p, q < \infty$.

We recall that in [4] an A-convex containing space $\mathcal{V}$ was exhibited for the pair $(L^{p_0}(L^{q_0}_*), L^{p_1}(L^{q_1}_*))$ when $0 < p_j, q_j \leq \infty$, $j = 0, 1$ (in [4] only the case where the parameters are finite is worked out, but the other cases follow the same pattern). From this space $\mathcal{V}$, an A-convex containing space $\mathcal{W}$ was constructed for the pair $(T^{p_0}_{q_0}, T^{p_1}_{q_1})$, whenever $0 < p_j < \infty$ and $0 < q_j \leq \infty$, $j = 0, 1$, in the following fashion: $f \in \mathcal{W}$ if $Tf \in \mathcal{V}$ and the quasi-norm of f in $\mathcal{W}$ is defined to be $\|f\|_{\mathcal{W}} = \|Tf\|_{\mathcal{V}}$.

Interpolating the operator $T\colon T^{p_j}_{q_j} \to L^{p_j}(L^{q_j}_*)$, $j = 0, 1$, as in [4, lemma 4], we arrive to de norm decreasing inclusion

$$(T^{p_0}_{q_0}, T^{p_1}_{q_1})_{[\theta],\mathcal{W}} \subset T^p_q, \tag{3}$$

whenever $0 < p_j, q_j < \infty$, $j = 0, 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Using the duality theorem for complex interpolation, we can now prove that equality holds if $1 < p_j, q_j < \infty$, $j = 0, 1$, [4, proposition 5]. We next prove this result by a different method.

Let F be a complex-valued measurable function over $\mathbf{R}^n \times \mathbf{R}^{n+1}_+$ shuch that it is locally integrable with respect to the first argument. Then we consider the function over $\mathbf{R}^{n+1}_+$ defined by

$$\Pi F(y,t) = \frac{1}{|B(y,t)|} \int_{B(y,t)} F(s; y, t)\, ds. \tag{4}$$

The expression ΠF makes sense if F is a simple function over $\mathbf{R}^n$ whose values are measurable functions over $\mathbf{R}^{n+1}_+$, ie. an expression of the form:

$$F(s; y, t) = \sum_{j=1}^{h} \psi_j(y,t)\chi_{E_j}(s),$$

where E_j are measurable subsets of $\mathbf{R}^n$ and ψ_j are measurable functions on $\mathbf{R}^{n+1}_+$. Therefore, (4) makes also sense if $F \in L^p(L^q_*)$ and $1 < p, q < \infty$.

The following informal remarks may be worth mentioning despite the fact that they will not be used in the sequel. If we consider the function on $\mathbf{R}^{n+1}_+$ defined by $M(y,t) = |B(y,t)|$ and the vector-valued function on $\mathbf{R}^n$

$$w(s) = \frac{1}{M}\chi_{\Gamma(s)},$$

we have, formally, a vector probability density on $\mathbf{R}^n$, since

$$\int_{\mathbf{R}^n} w(s)\, ds = 1,$$

where the last equation means that the integral is the identically equal to one function. Thus ΠF is, at least formally, the mean of F with respect to w, namely:

$$\Pi F = \int_{\mathbf{R}^n} w(s) F(s)\, ds.$$

Theorem 1. *Let $1 < p, q < \infty$. Then the operator Π is bounded from $L^p(L^q_*)$ into T^p_q. Furthermore, the composition $\Pi \circ T$ is the identity operator in T^p_q and thus T^p_q is a complemented subspace of $L^p(L^q_*)$.*

Proof. The fact that Π is a retract of T is clear. For the boundedness of Π, we can observe that under the dual pairing (1) Π is the adjoint operator of T and therefore it is bounded as claimed.

Theorem 2. *Let $0 \leq \theta \leq 1$ and $1 < p_j, q_j < \infty$, $j = 0, 1$, then*

$$(T^{p_0}_{q_0}, T^{p_1}_{q_1})_{[\theta]} = T^p_q,$$

where the relations

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

hold.

We observe that, since we are now in the Banach space case, we don't need any subscript $\mathcal{W}$ in the notation of our interpolation functor.

Proof of theorem 2. By (3), we only have to prove the inclusion

$$T^p_q \subset (T^{p_0}_{q_0}, T^{p_1}_{q_1})_{[\theta]}.$$

If $f \in T^p_q$, let $F = Tf \in L^p(L^q_*) = (L^{p_0}(L^{q_0}_*), L^{p_1}(L^{q_1}_*))_{[\theta]}$.

Let $\Phi \in \mathcal{F}(L^{p_0}(L^{q_0}_*), L^{p_1}(L^{q_1}_*))$ be an admissible function with $\Phi(\theta) = F$. Then $\Pi \circ \Phi \in \mathcal{F}(T^{p_0}_{q_0}, T^{p_1}_{q_1})$ is an admissible function for the pair of tent spaces whose value at θ is f. Thus the equality is proved.

We now pass to cover the general intermediate case.

Theorem 3. *Let $0 \leq \theta \leq 1$ and $0 < p_j, q_j < \infty$, $j = 0, 1$. If $\mathcal{W}$ is the above mentioned A-convex containing space for the pair $(T^{p_0}_{q_0}, T^{p_1}_{q_1})$, then*

$$(T^{p_0}_{q_0}, T^{p_1}_{q_1})_{[\theta],\mathcal{W}} = T^p_q,$$

if the relations

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

hold.

Proof. Again from (3), we only have to prove one inclusion. The idea is to reduce the proof to theorem 2 by a convex reduction argument. Let $M \geq 1$ be an integer such that $Mp_j, Mq_j > 1$, $j = 0, 1$. Let $f \in T^p_q$ be a simple function with compact support. We can find another simple function g such that $g^M = f$. Then $\|g\|^M_{T^{Mp}_{Mq}} = \|f\|_{T^p_q}$. By theorem 2, we know that

$$(T^{Mp_0}_{Mq_0}, T^{Mp_1}_{Mq_1})_{[\theta]} = T^{Mp}_{Mq}. \tag{5}$$

It follows that $g \in T^{Mp_0}_{Mq_0} \cap T^{Mp_1}_{Mq_1}$. We apply now a lemma of Stafney [13] and conclude that there is, for each $\epsilon > 0$, a function

$$G \in \mathcal{G}(T^{Mp_0}_{Mq_0}, T^{Mp_1}_{Mq_1})$$

that verifies $G(\theta) = g$ and $\|G\|_{\mathcal{H}} \leq (1+\epsilon)\|g\|_{[\theta]}$.

Set $F = G^M$. The function F has finite rank and is $T^{p_0}_{q_0} \cap T^{p_1}_{q_1}$-valued, $F \in \mathcal{G}(T^{p_0}_{q_0}, T^{p_1}_{q_1})$ and

$$\|F(z)\|_{T^{p_j}_{q_j}} = \|G(z)\|^M_{T^{Mp_j}_{Mq_j}},$$

$j = 0, 1$. Thus $\|F\|_{\mathcal{H}} = \|G\|^M_{\mathcal{H}}$, $F(\theta) = f$ and

$$\begin{aligned} \|f\|_{[\theta],\mathcal{W}} &\leq (1+\epsilon)^M \|g\|^M_{[\theta]} \\ &\leq (1+\epsilon)^M C^M \|g\|^M_{T^{Mp}_{Mq}} \\ &\leq (1+\epsilon)^M C^M \|f\|_{T^p_q}, \end{aligned} \tag{6}$$

where the constant C comes from the equivalence of norms in (5) and depends on M.

Since the class of all simple compactly supported functions f is dense in the tent spaces T^p_q, $0 < p, q < \infty$, the theorem is proved.

We now turn to the identification of the interpolation spaces with respect to the A-convex envelope.

In this respect, it is convenient to recall the complex interpolation method of Rivière [12], Cwikel, Milman and Sagher [8]. We will denote by $\|\cdot\|_\theta$ the quasi-seminorm introduced by these authors in the intersection of any complex quasi-Banach pair (X_0, X_1):

$$\|x\|_\theta = \inf\{\|g\|_{\mathcal{H}} : g \in \mathcal{G}(X_0, X_1) \quad g(\theta) = x\}.$$

If the quasi-Banach couple (X_0, X_1) has an A-convex containing space $\mathcal{U}$, then the inequality $\|x\|_{[\theta],\mathcal{U}} \leq \|x\|_\theta$ holds on the intersection and $\|\cdot\|_\theta$ is a quasi-norm. We next see a way of identifying the interpolation spaces with respect to the A-convex envelope.

Lemma 1. *Let (X_0, X_1) be a complex quasi-Banach couple with an A-convex containing space $\mathcal{U}$. Suppose that the quasi-norms $\|\cdot\|_{[\theta],\mathcal{U}}$ and $\|\cdot\|_\theta$ are equivalent on $X_0 \cap X_1$, then the identity*

$$I : (X_0 \cap X_1, \|\cdot\|_{[\theta],A}) \to (X_0 \cap X_1, \|\cdot\|_{[\theta],\mathcal{U}})$$

extends to a topological isomorphism between $(X_0, X_1)_{[\theta],A}$ and $(X_0, X_1)_{[\theta],\mathcal{U}}$, where A denotes the A-convex envelope of the sum space $X_0 + X_1$.

Proof. The continuous inclusion $J : X_0 + X_1 \hookrightarrow \mathcal{U}$ extends to a bounded operator $J_A : A \to \mathcal{U}$. The operator J_A maps X_j onto X_j with constant 1, $j = 0, 1$, (it is the identity on these spaces), so

$$J_A : (X_0, X_1)_{[\theta],A} \to (X_0, X_1)_{[\theta],\mathcal{U}},$$

also with constant 1.

On the other hand, we have the inequality $\|x\|_{[\theta],A} \leq \|x\|_\theta$ holding on $X_0 \cap X_1$, thus:

$$\|x\|_{[\theta],A} \leq \|x\|_\theta \leq C\|x\|_{[\theta],\mathcal{U}} \leq C\|x\|_{[\theta],A},$$

for each $x \in X_0 \cap X_1$. Hence we have proved that the quasi-norms $\|\cdot\|_{[\theta],A}$ and $\|\cdot\|_{[\theta],\mathcal{U}}$ are equivalent on $X_0 \cap X_1$ and using the density of the intersection in the complex interpolation spaces, the lemma follows.

Theorem 4. *Let $0 < p_j, q_j < \infty$, $j = 0, 1$, $0 \leq \theta \leq 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Then, the identity on $T^{p_0}_{q_0} \cap T^{p_1}_{q_1}$ extends to a topological isomorphism between $(T^{p_0}_{q_0}, T^{p_1}_{q_1})_{[\theta],A}$ and T^p_q.*

Proof. With the aid of the preceeding lemma, it is enough to prove that $\|\cdot\|_{[\theta],\mathcal{W}}$ is equivalent to $\|\cdot\|_\theta$ on $T^{p_0}_{q_0} \cap T^{p_1}_{q_1}$. To see this, we observe that the function F appearing in the proof of theorem 3 is of finite rank. Hence we could have written in (6):

$$\|f\|_\theta \leq C^M \|f\|_{T^p_q}.$$

Therefore, since $\|f\|_{T^p_q} \leq \|f\|_{[\theta],\mathcal{W}} \leq \|f\|_\theta$, the result follows for any simple compactly supported f and therefore for any $f \in T^{p_0}_{q_0} \cap T^{p_1}_{q_1}$. This completes the proof of the theorem.

We end this section by showing that, in some cases, we can claim that we have an isometric identification in theorems 3 and 4.

Theorem 5. *Let $0 < p_0, q_0 < \infty$, $0 \leq \theta \leq 1$ and $\lambda > 0$. Then*

$$(T^{p_0}_{q_0}, T^{\lambda p_0}_{\lambda q_0})_{[\theta],\mathcal{W}} = T^p_q,$$

isometrically, if $1/p = (1-\theta)/p_0 + \theta/(\lambda p_0)$ and $1/q = (1-\theta)/q_0 + \theta/(\lambda q_0)$. Furthermore, the identity on $T^{p_0}_{q_0} \cap T^{\lambda p_0}_{\lambda q_0}$ extends to a linear isometry between $(T^{p_0}_{q_0}, T^{\lambda p_0}_{\lambda q_0})_{[\theta],\mathcal{A}}$ and T^p_q.

Proof. Let f be a simple non-identically zero compactly supported function on $\mathbf{R}^{n+1}_+$. Define:

$$\Phi(z) = \frac{1}{\|f\|_{T^p_q}^{p\left(\frac{1}{\lambda p_0} - \frac{1}{p_0}\right)(z-\theta)}} |f|^{q\left(\frac{1}{\lambda q_0} - \frac{1}{q_0}\right)(z-\theta)} f.$$

It is clear that $\Phi \in \mathcal{G}(T^{p_0}_{q_0}, T^{\lambda p_0}_{\lambda q_0})$ and $\Phi(\theta) = f$, thus $\|f\|_\theta \leq \|\Phi\|_{\mathcal{H}}$. We observe that

$$p\left(\frac{1}{\lambda p_0} - \frac{1}{p_0}\right) = q\left(\frac{1}{\lambda q_0} - \frac{1}{q_0}\right) = C$$

and we can write

$$\Phi(z) = \left(\frac{|f|}{\|f\|_{T^p_q}}\right)^{C(z-\theta)} f.$$

We now estimate $\|\Phi\|_{\mathcal{H}}$.

$$|\Phi(it)| = \left(\frac{|f|}{\|f\|_{T^p_q}}\right)^{-C\theta} |f|,$$

but $-C\theta = -1 + (p/p_0)$, thus $|\Phi(it)| = |f|^{q/q_0} \|f\|_{T^p_q}^{1-(p/p_0)}$. From this, it follows:

$$\begin{aligned}\|\Phi(it)\|^{p_0}_{T^{p_0}_{q_0}} &= \|f\|_{T^p_q}^{(1-\frac{p}{p_0})p_0} \int_{\mathbf{R}^n} \left(\int_{\Gamma(x)} |f(y,t)|^q \frac{dy\,dt}{t^{n+1}}\right)^{p_0/q_0} dx \\ &= \|f\|_{T^p_q}^{(1-\frac{p}{p_0})p_0} \int_{\mathbf{R}^n} A_q f(x)^{\frac{q p_0}{q_0}}\, dx.\end{aligned}$$

Since $p_0/q_0 = p/q$, we get

$$\|\Phi(it)\|^{p_0}_{T^{p_0}_{q_0}} = \|f\|^{p_0 - p}_{T^p_q} \int_{\mathbf{R}^n} A_q f(x)^p\, dx = \|f\|^{p_0}_{T^p_q}.$$

In a similar fashion, $\|\Phi(1+it)\|_{T^{\lambda p_0}_{\lambda q_0}} = \|f\|_{T^p_q}$ and $\|\Phi\|_{\mathcal{H}} = \|f\|_{T^p_q}$. Therefore, we have proved that

$$\|f\|_{[\theta],\mathcal{W}} \leq \|f\|_\theta \leq \|\Phi\|_{\mathcal{H}} = \|f\|_{T^p_q} \leq \|f\|_{[\theta],\mathcal{W}}$$

and, by the density of all simple compactly supported functions, we arrive to the equality of the quasi-norms

$$\|\cdot\|_{[\theta],\mathcal{W}} = \|\cdot\|_\theta$$

on the intersection. If we observe the proof of lemma 1 in the case that the quasi-norms $\|\cdot\|_{[\theta],\mathcal{U}}$ and $\|\cdot\|_\theta$ are not only equivalent but equal, we will observe that the isometry between $(T^{p_0}_{q_0}, T^{\lambda p_0}_{\lambda q_0})_{[\theta],\mathcal{A}}$ and T^p_q is established.

3. Remarks on some extreme cases

We start by recalling that the identification

$$(T^{p_0}_\infty, T^{p_1}_\infty)_{[\theta]} = T^p_\infty,$$

where $1 < p_0, p_1 < \infty$, $0 \leq \theta \leq 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$, has been obtained in [1].

Lemma 2. *Let $1 < p_1, q_0, q_1 < \infty$ and $0 < \theta < 1$. Let $p = p_1/\theta$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, then*

$$(T^\infty_{q_0}, T^{p_1}_{q_1})_{[\theta]} = T^p_q.$$

Proof. We apply the duality theory of the tent spaces as well as the duality theorem of complex interpolation. We use the fact that $T^{p_1}_{q_1}$ is reflexive and theorem 3. We get

$$T^p_q = (T^{p'}_{q'})' = (T^1_{q_0'}, T^{p_1'}_{q_1'})'_{[\theta]} = (T^\infty_{q_0}, T^{p_1}_{q_1})_{[\theta]},$$

and the lemma is proved.

Proposition 1. *The formula*

$$(T^\infty_{q_0}, T^{p_1}_{q_1})_{[\theta]} = T^p_q,$$

is valid whenever $1 < q_0 < \infty$ and $1 \leq p_1, q_1 < \infty$, $0 < \theta < 1$ and p and q are determined as in lemma 2.

Proof. We just apply theorem 3, lemma 2 and Wolff's reiteration theorem [14].

The above argument cannot be directly applied to cover the case where $0 < p_1, q_1 < \infty$ because we don't have Wolff's reiteration theorem for the complex method in the quasi-Banach case. However Gomez and Milman obtained in [10] a Wolff type theorem which is valid for lattices. It can be applied joint with theorem 3 and lemma 2 to obtain the following:

Theorem 6. *Let $1 < q_0 < \infty$, $0 < p_1, q_1 < \infty$, $0 < \theta < 1$, $p = p_1/\theta$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, then the quasi-norms $\|\cdot\|_{T^p_q}$ and $\|\cdot\|_\theta$ are equivalent on $T^\infty_{q_0} \cap T^{p_1}_{q_1}$.*

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SOME RECENT GENERAL RESULTS IN INTERPOLATION THEORY

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1. INTRODUCTION

I think that most of the results which I will present here will be new for most participants, but the main aim of my talk is not so much to present new results, but rather to draw attention to some interesting research problems. All of these problems are related to the so called general theory of interpolation spaces, more specifically, to the construction and study of the properties of various interpolation methods.

The development of this theory is still far from complete. There are two important considerations which have motivated its development up till now and which should continue to do so in the future.

* First, the theory should give us a stable framework for calculations which before were obtained by good luck and ingenuity.

* Secondly, it should be able to give timely answers to new questions which constantly arise in applications.

At the same time, specific calculations and applications are a powerful stimulus for the development of the theory. Indeed, its successful development is impossible without the prior accumulation of a "critical mass" of concrete results. Such a "critical mass" had collected at the end of the fifties, and it led to the glorious period of formation and development of the theory during the years 1959-1966. Since then some important papers have appeared from time to time in this field but of course their quantity (not quality) cannot be compared with that powerful stream of papers dealing with calculations and